PROJECTED WRITTEN NOTES FROM THE M325K LECTURE

on Tuesday, January 30, 2024, on UNIVERSAL CONDITIONALS, Combining Quantifiers, and Universal Arguments.

CLASS # 5

Negations of Universal Statements and of Existential Statements.

The Negation of the universal Statement

"$\forall x \in D, P(x)$"

is

"$\sim(\forall x \in D, P(x))$" $\equiv$ "$\exists x \in D$ such that $\sim P(x)$"

The Negation of the existential Statement

"$\exists x \in D$ such that $P(x)$"

is

"$\sim(\exists x \in D$ such that $P(x))$" $\equiv$ "$\forall x \in D, \sim P(x)$".

Statements	It's Negation
S: All politicians are honest.	~S: There is a politician who is not honest.
	NOT ALL POLITICIANS ARE HONEST.

WRONG NEGATIONS:

ALL POLITICIANS ARE NOT HONEST.

SOME POLITICIANS ARE HONEST.

If there is a case where both Statements are True or both Statements are false, then they are not really negations of each other.

Related Universal Conditionals

ORIGINAL UNIVERSAL CONDITIONAL

For all real numbers x, if $x<0$, then $x^2>0$.

UNIVERSAL CONVERSE

For all real number x, if $x^2>0$, then $x<0$.

UNIVERSAL INVERSE $(x \not< 0)$

For all real numbers x, if $x \geq 0$, then $x^2 \leq 0$.

UNIVERSAL CONTRAPOSITIVE

For all real numbers x, if $x^2 \leq 0$, then $x \geq 0$.

These can be worded using "sufficient", "necessary", "only if", etc..

Exercise: Rephrase Statement S without using "sufficient". Note: f is a function, $f: \mathbb{R} \to \mathbb{R}$.

S: "Being continuous is not a sufficient condition for a function to be differentiable."

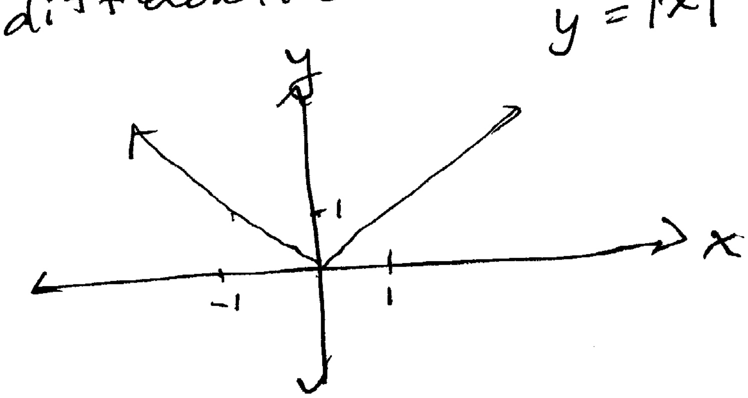
Sol'n: $S \equiv \sim T$

where T is "Being continuous is a sufficient condition for a function f to be differential"

T: If a function is continuous, then it is differential.

T: (For all functions f, if f is continuous then f is differentiable")

$S \equiv \sim T$: There exists a function f such that f is continuous and f is not differentiable.



$\sim(p \to q) \equiv p \wedge \sim q$

Combining Quantifiers: $\forall$ and $\exists$

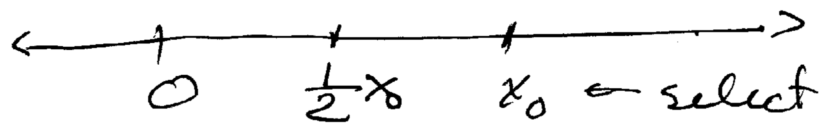
let $Q(x,y) =$ "$y < x$"

(1) For every positive real number x, there exists a positive real number y such that $y < x$

T

STATEMENT FORM: $\forall x \in D_1,\ \exists y \in D_2$ such that $Q(x,y)$

The value of y depends on the chosen value of x.



x_0 ← select this anywhere such that $x_0 > 0$.

let $y = \frac{1}{2}x_0$

$0 < y = \frac{1}{2}x_0 < x_0$

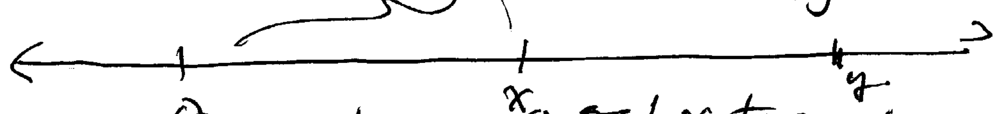
(2) There exists a positive real number x such that, for every positive real number y, $y < x$.

F

STATEMENT FORM: $\exists x \in D_1$ such that, $\forall y \in D_2$, $Q(x,y)$.

Every positive real number is $< x_0$?

let $y = 2x_0$



No!

x_0 ← Locate a value of x that exists like this.

UNIVERSAL STATEMENTS IN ARGUMENTS

UNIVERSAL Instantiation = { If a Predicate $P(x)$ is true for all x in a domain D and if x_0 is a particular element in Domain D, Then $P(x_0)$ is a True Statement.

UNIVERSAL MODUS PONENS

$$\forall x \in D,\ P(x) \to Q(x)$$

$$x_0 \in D \text{ and } P(x_0)$$

$\therefore\ Q(x_0)$ by Universal Ponens.

<u>Argument</u> If m and n are any odd integers, then $m \times n$ is an odd integer.

Universal Instation. — If 7 and 5 are odd integers, then 7×5 is odd.

MODUS PONES — 7 and 5 are odd integers

$\therefore\ 7 \times 5$ is odd

UNIVERSAL MODUS PONENS

Step 1:	For all real numbers x, $x^1 = x$.	universal truth
	r is a particular real number.	particular instance
	$\therefore r^1 = r$.	conclusion
Step 2:	For all real numbers x and all integers m and n, $x^m \cdot x^n = x^{m+n}$.	universal truth
	r is a particular real number and $k+1$ and 1 are particular integers.	particular instance
	$\therefore r^{k+1} \cdot r^1 = r^{(k+1)+1}$.	conclusion

Both arguments are examples of universal instantiation.

Universal Modus Ponens

The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called *universal modus ponens.*

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Universal Modus Ponens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
$\therefore Q(a)$.	$\therefore$ a makes $Q(x)$ true.

Note that the first, or major, premise of universal modus ponens could be written "All things that make $P(x)$ true make $Q(x)$ true," in which case the conclusion would follow by universal instantiation alone. However, the if-then form is more natural to use in the majority of mathematical situations.

Example 2.4.1 Recognizing Universal Modus Ponens

Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

If a number is even, then its square is even.
k is a particular number that is even.
$\therefore k^2$ is even.

Solution The major premise of this argument can be rewritten as

$$\forall x, \text{ if } x \text{ is even then } x^2 \text{ is even.}$$

Let $E(x)$ be "x is even," let $S(x)$ be "x^2 is even," and let k stand for a particular number that is even. Then the argument has the following form:

$\forall x$, if $E(x)$ then $S(x)$.
$E(k)$, for a particular k.
$\therefore S(k)$.

Of course, the actual proof that the sum of even integers is even does not explicitly contain the sequence of arguments given above. (Heaven forbid!) And, in fact, people who are good at analytical thinking are normally not even conscious that they are reasoning in this way. But that is because they have absorbed the method so completely that it has become almost as automatic as breathing.

Universal Modus Tollens

Another crucially important rule of inference is ***universal modus tollens.*** Its validity results from combining universal instantiation with modus tollens. Universal modus tollens is the heart of proof of contradiction, which is one of the most important methods of mathematical argument.

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Universal Modus Tollens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$, for a particular a.	a does not make $Q(x)$ true.
$\therefore \sim P(a)$.	$\therefore$ a does not make $P(x)$ true.

Example 2.4.3 Recognizing the Form of Universal Modus Tollens

Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All human beings are mortal.
Zeus is not mortal.
$\therefore$ Zeus is not human.

Solution The major premise can be rewritten as

$$\forall x, \text{ if } x \text{ is human then } x \text{ is mortal.}$$

Let $H(x)$ be "x is human," let $M(x)$ be "x is mortal," and let Z stand for Zeus. The argument becomes

$$\forall x, \text{ if } H(x) \text{ then } M(x)$$
$$\sim M(Z)$$
$$\therefore \sim H(Z).$$

This argument has the form of universal modus tollens and is therefore valid. ■

Example 2.4.4 Drawing Conclusions Using Universal Modus Tollens

Write the conclusion that can be inferred using universal modus tollens.

The conclusion "Felix is a human being" is true in the first case but not in the second (Felix might, for example, be a cat). Because the conclusion does not necessarily follow from the premises, the argument is invalid. ■

The argument of Example 2.4.6 would be valid if the major premise were replaced by its converse. But since a universal conditional statement is not logically equivalent to its converse, such a replacement cannot, in general, be made. We say that this argument

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Converse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
$\therefore P(a)$. ← invalid conclusion	$\therefore$ a makes $P(x)$ true. ← invalid conclusion

cally equivalent to its inverse. But it is not, and the argument form is invalid. We say that it exhibits the inverse error. You are asked to show the invalidity of this argument form in the exercises at the end of this section.

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Inverse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$, for a particular a.	a does not make $P(x)$ true.
$\therefore \sim Q(a)$. ← invalid conclusion	$\therefore$ a does not make $Q(x)$ true. ← invalid conclusion

Example 2.4.7 An Argument with "No"

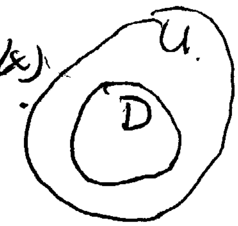
Use diagrams to test the following argument for validity:

No polynomial functions have horizontal asymptotes.
This function has a horizontal asymptote.
$\therefore$ This function is not a polynomial function.

Solution A good way to represent the major premise diagrammatically is shown in Figure 2.4.6, two disks—a disk for polynomial functions and a disk for functions with horizontal asymptotes—that do not overlap at all. The minor premise is represented by placing a dot labeled "this function" inside the disk for functions with horizontal asymptotes.

Direct Universal Statements have equivalent wording as Universal Conditional Statements.

Direct Universal	EQUIVALENT UNIVERSAL CONDITIONAL
$\forall x \in D,\ Q(x)$.	$\forall x \in U$, if $x \in D$, then $Q(x)$.
All worthy Goals should be pursued.	For any goal, if it is worthy, then it should be pursued.



Validity in Universal Arguments

All toddlers fidget.
Sue does not fidget.

$\therefore$ Sue is not a toddler.

$\forall x \in$ People, if $Tod(x) \rightarrow Fid(x)$.
$\sim Fid(Sue)$

$\therefore \sim Tod(Sue)$

This Argument is Valid by UNIVERSAL MODUS TOLLENS.

Ex:

No Good student failed this course

Herb is not a good Student

∴ Herb failed this Course.

Invalid by the UNIVERSAL INVERSE ERROR.

$\forall x \in$ Students

$GS(x) \to \sim FC(x)$

$\sim GS(\text{Herb})$

$\therefore\ FC(\text{Herb})$

$\sim(\sim FC(\text{Herb}))$

$$\left.\begin{array}{l} p \to q \\ \sim p \\ \hline \therefore \sim q \end{array}\right\} \text{Inverse ERROR}$$